# COMPARISON THEOREMS FOR THE EQUATIONS OF NONSTATIONARY FILTRATION

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In the nonstationary filtration of a compressible fluid, the distribution of the pressure is described by a quasi-linear equation of parabolic type. A number of exact (similarity) solutions of this equation are known, corresponding to special classes of initial and boundary conditions. To effect a solution, approximate methods are generally used which, as a rule, do not have a rigorous foundation and which do not yield error estimates.

In a number of cases one may obtain an approximate solution which is very close to the exact solution by taking for the distribution of the mass velocities of filtration those found from the solution of the linearized problem.

This is true for both filtration that follows Darcy's law [1] as well as for nonlinear filtration [2]. The linearization is equivalent to replacing the variable coefficients in the equations of filtration by constants. Therefore, the error in linearization, like the error in certain other approximate methods, may be estimated by studying the dependence of the solution on the coefficients of the equations. Comparison theorems which determine the character of this dependence were obtained by Pirverdian for similarity solutions [3 and 4].

In the present paper we examine comparison theorems for one-dimensional filtration equations without the assumption of similarity. In addition, we examine estimates of the maximum principle type which allow one to determine bounds on the change of the solution as a function of initial and boundary conditions. The results which are obtained are used to estimate the accuracy of the linearized solution of the equations of gas filtration.

**1. The dependence of the solutions on the initial and boundary conditions.** The one-dimensional filtration of a compressible fluid in a uniform layer is described by the system

$$\frac{k\rho}{\mu}\frac{\partial p}{\partial x} = -\varphi(j), \qquad \frac{\partial (m\rho)}{\partial t} = -\frac{1}{x^s}\frac{\partial}{\partial x}(x^s j) \tag{1.1}$$

Here $p$ is the pressure, $\rho$ the density, $\mu$ the viscosity of the fluid, $k$ the permeability, $m$ the porosity of the rock, $j$ the mass velocity of filtration, and $s+1$ the dimension of the space.

The function $\varphi$ gives the filtration law and in its physical meaning increases monotonously and is odd. For filtration in accordance with Darcy's law $\varphi(j) \equiv j$.

We introduce new variables: the Leibenzon function $P(p)$ and $q = -x^s J$ (instantaneous mass flow through a unit angle of the coordinate surface $x$ = const). In the new variables, the system (1.1) takes on the form

$$(A) \qquad \frac{\partial P}{\partial x} = \varphi(q / x^s) \qquad \left(P(p) = \int_0^p \frac{k\rho}{\mu}\, dp\right)$$

$$\frac{\partial P}{\partial t} = \varkappa(P) \frac{1}{x^s} \frac{\partial q}{\partial x}, \qquad \varkappa(P) = \left(\frac{d(m\rho)}{dP}\right)^{-1} > 0 \qquad (1.2)$$

One may eliminate $q$ from Equations (1.2). The isothermal filtration of a perfect gas which obeys Darcy's law was investigated in [5] by means of the equation that is obtained. For this equation the maximum principle and the monotone dependence on initial and boundary conditions were proved. Later, these results were generalized in [6], where the existence and uniqueness of the solution of the basic problems was proved and where the properties of solutions of the equation of nonstationary gas filtration were investigated. If one assumes a sufficient smoothness of the solution, then the maximum principle and the monotone dependence on initial and boundary conditions may also be proved for the system of equations (1.2).

The proof may be carried out by methods which are applied in [7] for linear equations of parabolic type.

For the problem which is studied in the rectangle $D$ $(0 < a < x < b,\ 0 < t < T)$, the following assertions are true (by $\Gamma$ we denote the boundary of the rectangle $D$ without the upper base $t = T$, $a < x < b$):

1.1. If $P|_\Gamma \geqslant 0$, then $P \geqslant 0$ everywhere in $D^\circ$ $(a \leqslant x \leqslant b,\ 0 \leqslant t \leqslant T)$

1.2. If $q|_\Gamma \geqslant 0$, the $q \geqslant 0$ everywhere in $D^\circ$.

1.3. Let the function $\psi$, the inverse of $\varphi$, satisfy the Lipschitz condition, and for $y \geqslant \eta > 0$ ($\eta$ is an arbitrary number) $\varphi'(y) \leqslant N_\eta < \infty$. Then the maximum principle holds for the function $P$: if $m \leqslant P|_\Gamma \leqslant M$, then $m \leqslant P \leqslant M$ everywhere in $D^\circ$.

1.4. Let the function $\varkappa(P)$ satisfy the Lipschitz condition. We assume that the functions $P_1$ and $P_2$ satisfy the system (1.2) and the condition

$$(P_1 - P_2)_\Gamma \geqslant 0 \qquad (1.3)$$

Then if

$$\left|\frac{\partial P_2}{\partial t}\right| < Mt^{-r}, \qquad r < 1$$

then everywhere in $D^\circ$

$$P_1 - P_2 \geqslant 0 \qquad (1.4)$$

In the inequalities (1.3) and (1.4) the signs may be given opposite senses.

1.5. If the coefficients of the system (1.2) satisfy the conditions

$$0 < \eta \leqslant \varphi'(j) \leqslant N_1 < \infty, \qquad |\varkappa'(P)| \leqslant N < \infty$$

then from $m \leqslant q|_\Gamma \leqslant M$ it follows that $m \leqslant q \leqslant M$ everywhere in $D^\circ$.

The theorems which have been introduced allow one to crudely estimate the solutions in accordance with the initial and boundary conditions.

**2. Comparison theorems.** We examine the functions $P_1$ and $q_1$ satisfying a system of the same form as system $(A)$ of Equations (1.2), but with certain, other coefficients $\varkappa_1(P_1)$. We shall call this system, system $(B)$ Subtracting from the equations of system $(B)$ the corresponding equations of system $(A)$, we obtain

$$\frac{\partial(P_1 - P)}{\partial x} = \varphi(q_1 / x^s) - \varphi(q / x^s) \qquad (2.1)$$

$$\frac{\partial(P_1 - P)}{\partial t} = \frac{\varkappa_1(P_1)}{x^s} \frac{\partial(q_1 - q)}{\partial x} + \frac{\varkappa_1(P_1) - \varkappa(P)}{x^s} \frac{\partial q}{\partial x} \qquad (2.2)$$

We shall find conditions for which $P_1 - P_2 \geqslant 0$ everywhere in the closed rectangle $D^\circ$. To do this we examine the function $U_1 = (P_1 - P) e^{-\alpha t}$. We assume that it has a minimum at a certain point $(x_1, t_1) \in D_*$ (by $D_*$ we denote the rectangle $D^\circ$ without the $\Gamma$ part of the boundary). Using Equations (2.1) and (2.2), it is easy to convince oneself that at the minimum point

$$\frac{\partial U_1}{\partial t} = \frac{\varkappa_1 (P_1)}{x^s \varphi'(q / x^s)} \frac{\partial^2 U_1}{\partial x^2} + e^{-\alpha t} \frac{\varkappa_1 (P_1) - \varkappa (P_1)}{x^s} \frac{\partial q}{\partial x} + \\ + \frac{\varkappa (P_1) - \varkappa (P)}{x^s} e^{-\alpha t} \frac{\partial q}{\partial x} - \alpha U_1 \tag{2.3}$$

We assume that in $D_*$

$$0 \leqslant \partial q / \partial x \leqslant N < \infty, \qquad \varkappa (P) \leqslant \varkappa_1 (P) \tag{2.4}$$

and that the function $\varkappa(P)$ satisfies the Lipschitz condition

$$| \varkappa (P_1) - \varkappa (P_2) | \leqslant C | P_1 - P_2 | \tag{2.5}$$

We take $\alpha > CN/a^s$. Using Equation (2.3) and the fact that at the minimum point $\partial U_1 / \partial t \leqslant 0$, $\partial^2 U_1 / \partial x^2 \geqslant 0$, it may be verified that $U_1 (x_1, t_1) \geqslant 0$.

Therefore the following assertion is valid.

T h e o r e m 2.1. If conditions (2.4), (2.5) and

$$(P_1 - P) |_\Gamma \geqslant 0 \tag{2.6}$$

are satisfied, then $P_1 - P \geqslant 0$ everywhere in $D^\circ$.

Indeed, in the opposite case the function $U_1$ has in $D_*$ a negative minimum for arbitrary $\alpha$, which contradicts what was proved above.

If in the second inequality (2.4), and likewise in inequality (2.6), the signs are changed, then the assertion of the theorem, clearly, also changes in the opposite direction. The same occurs if the first condition (2.4) is changed to

$$- \infty < - N \leqslant \partial q / \partial x \leqslant 0 \tag{2.7}$$

Instead of the requirement of boundedness of the derivative $\partial q/\partial x$, one may require that $| \partial q / \partial x | < N_1 t^{r-1}$, $r > 0$. The proof then carries through; it is sufficient to change $\exp(-\alpha t)$ into $\exp(-\alpha t^r)$.

Theorem 2.1 has a simple physical interpretation. If $\partial q / \partial x \geqslant 0$, then the pressure at every point of the layer increases, as follows from Equation (1.2). The magnitude of the total increase in pressure from the beginning of the process is determined by the initial and boundary conditions and by the speed of levelling of the perturbations, growing with an increase in the latter. Therefore, under the assumptions that have been made, an increase of $\varkappa(P)$ which leads to speeding up of the levelling, increases the pressure increment.

In many problems of filtration theory the flow rather than the pressure is prescribed on the boundary of the region. Using the same method of proof, one may in this case also obtain some comparison theorems. In this case, since the flow is a function of $\partial P/\partial x$, it is necessary to differentiate the original equation and to add some restrictions on the derivatives of the functions $\varkappa(P)$ and $\varphi(J)$.

We shall compare system $(C)$, determined by the functions $\varphi_1(J_1)$ and $\varkappa_1(P_1)$, with the system $(A)$. Let the functions $P_1$ and $q_1$ satisfy this system.

Differentiating the second equation of system $(A)$ with respect to $x$ and using the first equation of this system, we obtain

$$\varphi' \left(\frac{q}{x^s}\right) \frac{\partial q}{\partial t} = \varkappa (P) \left(\frac{\partial^2 q}{\partial x^2} - \frac{s}{x} \frac{\partial q}{\partial x}\right) + \varkappa'(P) \varphi \left(\frac{q}{x^s}\right) \frac{\partial q}{\partial x} \tag{2.8}$$

If one writes an analogous equation based on the system $(C)$ and subtracts from it Equation (2.8), then after some transformations one finds

$$\varphi_1'(q_1/x^s)\frac{\partial(q_1-q)}{\partial t}+[\varphi_1'(q_1/x^s)-\varphi'(q/x^s)]\frac{\partial q}{\partial t}=$$
$$=\varkappa_1(P_1)\left[\frac{\partial^2(q_1-q)}{\partial x^2}-\frac{s}{x}\frac{\partial(q_1-q)}{\partial x}+\frac{\varkappa_1'(P_1)}{\varkappa_1(P_1)}\varphi_1(q_1/x^s)\times\right.$$
$$\left.\times\frac{\partial(q_1-q)}{\partial x}\right]+\varkappa_1(P_1)\left[\frac{\varkappa_1'(P_1)}{\varkappa_1(P_1)}\varphi_1(q_1/x^s)-\right.$$
$$\left.-\frac{\varkappa'(P)}{\varkappa(P)}\varphi(q/x^s)\right]\frac{\partial q}{\partial x}+\frac{\varkappa_1(P_1)-\varkappa(P)}{\varkappa(P)}\varphi'(q/x^s)\frac{\partial q}{\partial t} \qquad (2.9)$$

We assume that

$$\partial q/\partial t\geqslant 0,\qquad \partial q/\partial x\leqslant 0,\qquad q\geqslant 0 \qquad (2.10)$$

$$0<\varepsilon\leqslant\varphi_1'(q/x^s)\leqslant\varphi'(q/x^s)\leqslant N_\varphi<\infty \qquad (2.11)$$

$$\varkappa_1(P_1)\geqslant\varkappa(P),\quad \varkappa_1'(P_1)/\varkappa_1(P_1)\leqslant\varkappa'(P)/\varkappa(P),\quad 0\leqslant\varkappa'(P)/\varkappa(P)\leqslant N_\varkappa \qquad (2.12)$$

and that the function $\varphi'(j)$ satisfies the Lipschitz condition

$$|\varphi'(j_1)-\varphi'(j_2)|\leqslant C_\varphi|j_1-j_2| \qquad (2.13)$$

It is sufficient to require that the conditions (2.10) to (2.13) be fulfilled for those values of the arguments which may be encountered in the problem at hand. The range of variation of these variables may be estimated beforehand by means of the theorems in Section 1. We assume likewise that $\partial q/\partial t$ and $\partial q/\partial x$ are bounded or that they have singularities of the form $At^{r-1}$

T h e o r e m 2.2. Under the conditions (2.10) to (2.13) and the consequences of $|q_1-q|_\Gamma\geqslant 0$ it follows that $q_1\geqslant q$ everywhere in $D^\circ$.

For a proof it is sufficient to consider the function

$$U_2=(q_1-q)\exp(-\alpha t^r)$$

and by means of the relation (2.9) to verify that it cannot have a negative minimum in $D_*$.

Using varius combinations of the signs in inequalities (2.10) to (2.12), one may obtain various modifications of Theorem 2.2.

The conditions of Theorem 2.2 contain a large number of requirements placed on the functions entering into the system $(A)$ of equations and the solution of this system.

By means of examples it is not difficult to show that the requirements of the monotonous behavior of the solutions and the requirements $\varkappa_1(P_1)\geqslant\varkappa(P)$ and $\varphi_1'(j)\leqslant\varphi'(j)$ are essential. At the same time, the requirement $\varkappa_1'(P_1)/\varkappa_1(P_1)\leqslant\varkappa'(P)/\varkappa(P)$ which appears because of the method of proof, turns out to be excessively restrictive and not satisfied in a number of important cases. Instead of this requirement it would have been natural to require the smallness of $\varkappa_1'(P_1)/\varkappa_1(P_1)$ and $\varkappa'(P)/\varkappa(P)$.

A sufficient result for a number of applications is obtained if one considers the function

$$U_3=(q_1V_{\beta_1}-qV_{-\beta})e^{-\alpha t},\qquad \beta_1\geqslant 0,\quad \beta\geqslant 0 \qquad (2.14)$$

Here $V_\beta(x)$ denotes the solution to Equation

$$\frac{dV_\beta}{dx}=\frac{\beta}{x^s}V_\beta,\qquad V_\beta(a)=1 \qquad (2.15)$$

Obviously, the function $V_\beta(x)$ is positive, monotonously increasing for $\beta>0$ and decreasing for $\beta<0$.

By the same means one may show that $U_3$ does not have a negative minimum in $D_*$ when conditions (2.10) and the following conditions are fulfilled

$$0 < \varepsilon \leqslant \varphi_1'(j_1) \leqslant \varphi'(j) \leqslant E < \infty \qquad \text{for} \quad j \geqslant j_1 \tag{2.16}$$

$$\varkappa_1'(P_1)/\varkappa_1(P_1) < N_1, \qquad \varkappa'(P)/\varkappa(P) > -N$$

$$\varkappa_1(P_1) \geqslant \varkappa(P) \geqslant \eta\varkappa_1(P_1), \qquad \eta > 0 \tag{2.17}$$

and when $\beta_1$ and $\beta$ are chosen from conditions

$$\beta_1 = \max(\tfrac{1}{2} N_1 E Q_1, 0), \qquad Q_1 = \max q_1|_\Gamma \tag{2.18}$$

$$\beta = \max(\tfrac{1}{2} NEQ, 0, \beta^*), \qquad Q = \max q|_\Gamma \tag{2.19}$$

Here

$$\beta^* = \sqrt{\tfrac{1}{4} s^2 b^{2s-2} + \eta^{-1}\beta_1^2 - \eta^{-1}\beta_1 s b^{s-1}} - \tfrac{1}{2} s b^{s-1} \qquad (\beta_1 \geqslant s b^{s-1})$$

$$\beta^* = 0 \qquad (\beta_1 < s b^{s-1}) \tag{2.20}$$

T h e o r e m 2.3 . If conditions (2.10), (2.16),(2.17) and

$$(q_1 V_{\beta_1+\beta} - q)|_\Gamma \geqslant 0 \tag{2.21}$$

are satisfied for $\beta_1$ and $\beta$ , determined by (2.18) to (2.20), then $(q_1 V_{\beta_1+\beta} - q) \geqslant 0$ everywhere in $D^\circ$ .

In analogy with the above, one may obtain various modifications of Theorem 2.3 by changing the signs of the inequalities in conditions (2.10), (2.16) and (2.17).

In the use of the comparison theorem, the monotonous behavior of at least one of the comparison solutions is required. In the elastic regime for linear filtration $\varkappa(P)$ = const , $\varphi(j) \equiv j$ , so that the system $(A)$ turns out to be linear with constant coefficients. For the derivative solutions of such a system, the maximum principle is valid since the derivatives likewise satisfy an equation of parabolic type.

For the general case of system $(A)$ one may prove analogous assertions.

T h e o r e m 2.4 . Let conditions (1.5) be satisfied and likewise let the conditions

$$\partial q(x, 0)/\partial x \leqslant 0, \qquad q(a, t) = \Phi_a(t) \geqslant 0, \qquad q(b, t) = \Phi_b(t) \leqslant 0$$

$$\Phi_a'(t) \geqslant 0, \qquad \Phi_b'(t) \leqslant 0$$

be satisfied.

Then $\partial q/\partial x \leqslant 0$ everywhere in $D^\circ$.

The assertions obtained by reversing the signs in the inequalities are likewise valid.

**3. Example of the application of the comparison theorem.** Pirverdian estimated the accuracy of certain approximate methods by means of comparison theorems that he obtained in [3 and 4]. Using the theorems of Section 2, one may obtain further results. In particular, one may estimate the error caused by the linearization of the equations of gas filtration for plane radial flow.

We consider the plane radial influx of a gas into a hole of radius $a$ in a layer of radius $A$. We use a two term filtration law

$$\varphi_1(j) = j + \gamma j^2, \qquad j \geqslant 0, \qquad \gamma = \text{const} \geqslant 0 \tag{3.1}$$

We assume that the function $\varkappa_1(P_1)$ increases monotonously as is the case for a perfect gas $\varkappa_1'(P_1) \geqslant 0$. Let there be no initial motion in the layer

$$q_1(x, 0) = 0, \qquad P_1(x, 0) = P_0 \tag{3.2}$$

and let the influx into the hole start at time $t = 0$ with the flow

$$q_1(a, t) = \Phi(t), \qquad \Phi'(t) \geqslant 0 \tag{3.3}$$

whereby $\Phi(t)$ very rapidly attains the constant value $\Phi_0$. Such a problem is of fundamental importance for the establishment of methods of studying a layer by observing nonstationary influx. We assume that the far boundary of the layer is impermeable

$$q_1(A, t) = 0 \tag{3.4}$$

For sufficiently large radius of the layer $A$, condition (3.4) holds only for extremely large times, and in the initial interval may be replaced by another condition.

At the same time we consider the filtration of an elastic fluid for which $\varkappa_1 = \varkappa_1(P_0) = \text{const}$, $\varphi(j) \equiv j$ under the same initial and boundary conditions. By virtue of Theorem 2.4, the derivative $\partial P_1 / \partial t \leqslant 0$, so that $P_1 \leqslant P_0$. Therefore

$$\varkappa_1(P_1) \leqslant \varkappa_1(P_0) = \varkappa, \qquad \varkappa_1'(P_1) \geqslant 0, \qquad \varkappa'(P) = 0$$

$$\varphi_1'(j) = 1 + 2\gamma j \geqslant 1 = \varphi'(j) \tag{3.5}$$

and by virtue of Theorem 2.2

$$q_1(x, t) \leqslant q(x, t)$$

For sufficiently large $t$ the following formula holds for $q(x, t)$

$$q(x, t) = \Phi_0 \exp(-x^2/4\varkappa t) \qquad \text{for} \quad a^2/\varkappa \ll t \ll A^2/\varkappa \tag{3.6}$$

In the sequel we consider just those times for which Formula (3.6) is applicable. By means of the bound (3.5), the filtration law (3.1), and the first equation of system $(A)$ we obtain

$$P_1 \geqslant P_0 - \int_x^\infty \left[\frac{q_1(x, t)}{x} + \gamma \frac{q^2(x, t)}{x^2}\right] dx - \frac{\Phi_0 t}{\pi A^2 m \varkappa (P_1(a, t))} \tag{3.7}$$

We note that the last term in (3.7) is usally negligibly small.

Now we may obtain, although crudely, a bound from below for $q_1(x,t)$ and from above for $P_1$. In fact, for the interval of time $t \leqslant T$

$$q_1 \leqslant \Phi_0, \qquad \varkappa_1(P_1) \geqslant \varkappa_1(P_1(a, T))$$

We set

$$\varkappa = \varkappa_* = \varkappa(P_1(a, T)), \qquad \varphi = (1 + 2\gamma\Phi_0/a)\, j \tag{3.8}$$

and denote by $q_*$ the solution of the problem with $\varkappa$ and $\varphi$ determined by relations (3.8) and the previous boundary conditions. Theorem 2.2 is applicable to the functions $q_1$ and $q_*$. Choosing $\beta_1$ and $\beta$ as indicated above, we have

$$(x/a)^{-\beta-\beta_1} q_*(x, t) \leqslant q_1(x, t) \tag{3.9}$$

(in the case considered $s = 1$ and $V_\beta = (x/a)^\beta$).

For $q_*$ one may again indicate an explicit solution which gives with sufficient accuracy the expression

$$q_*(x, t) = \Phi_0 \exp\left[-\frac{x^2}{4\varkappa_*}\left(1 + 2\frac{\gamma\Phi_0}{a}\right)\right] \tag{3.10}$$

For the usual ratio of parameters $\beta_1 \sim 10^{-2}$. Hence $\beta$ may be set equal to zero, and for $q_1$ one obtains the bound

$$\left(\frac{x}{a}\right)^{-\beta_1} \exp\left(-\frac{x^2(1 + 2\gamma\Phi_0/a)}{4\varkappa_* t}\right) \leqslant \frac{q_1(x, t)}{\Phi_0} \leqslant \exp\left(-\frac{x^2}{4\varkappa t}\right) \tag{3.11}$$

(As before, the radius of the layer $A$ is assumed to be very large). Using the bound (3.11), one may show that for sufficiently large $t$ the following formula holds

$$(P_0 - P_1) / \Phi_0 \sim i \ln t + \gamma\Phi_0 / a + C \qquad (3.12)$$

The constant $C$ may contain terms which weakly (logarithmically) depend on $\Phi_0$, while the angular coefficient is determined by Formula

$$i = {}^1/_2 \, [1 - {}^1/_4 \, \beta_1 \ln (A / a) \qquad (3.13)$$

with a relative accuracy not worse than $\frac{1}{4}\beta_1 \ln (A/a)$. The quantity $A$ entered into (3.13) only because of the requirement $t \ll A^2 / \varkappa$, hence it may be replaced by another, arbitrary quantity which satisfies the same inequality for all the times under consideration. For the usual values of parameters the accuracy of Formula (3.13) is about one percent.

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